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A mixed boundary-value problem for the wave equation in a stratified medium for high-frequency oscillations☆

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Abstract

A boundary-value problem for the wave equation in a stratified medium with mixed boundary conditions on the boundary in the case of high oscillation frequencies is considered. The Helmholtz equation for a velocity function increasing monotonically with depth is investigated. The problem is reduced to an integral equation in the high-frequency approximation, and an explicitly smooth term of its asymptotic solution is constructed.

Several effective classical methods of solving boundary-value problems with mixed boundary conditions for linear partial differential equations exist at high frequencies (for example, the ray method and the WKB method in Refs. 1,2). As a rule, these analytical approaches operate with differential equations and are not suitable for investigating boundary-value problems. In some cases they can be applied to ordinary (unmixed) boundary-value problems. A good example of such applications in underwater acoustics can be found in well-known monographs (see, for example, Refs. 1–3).

The application of integral transformations to problems with mixed boundary conditions usually leads to the separation of variables, as in the case of equations with constant coefficients. But this does not provide any appreciable simplification of the problem, since the ordinary differential equation that is obtained after this separation cannot be solved analytically. An alternative approach, the so-called method of mode expansions, also cannot be applied in the case of non-constant coefficients (see Ref. 3).

The main aim of this paper is to construct an effective analytical method of solving the mixed boundary-value problem in the case of an acoustic medium, when the coefficient, related to the velocity of sound in the medium, is non-constant with depth. By using a Fourier transformation of the horizontal variable, the problem is reduced to an ordinary differential equation with variable coefficients. This equation has a point of rotation,[2–6] and hence the well-known WKB method cannot be applied to it. The solution is presented in terms of an Airy function. The problem is then reduced to a hypersingular integral equation. A brief review of different methods of solving such equations is given and it is shown that the well-known approach employing the method of boundary waves, previously developed solely for regular equations, can also be extended to hypersingular equations. This enables the principal asymptotic term of the expansion of the solution to be represented in explicit form.

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1. Mathematical formulation of the problem and the use of a Fourier transformation

Consider a stratified acoustic half-plane $z>0$, $|x|<\infty$ with a time-harmonic wave process

$$\tilde{p}(x,z,t) = \exp(-i\omega t)p(x,z) \tag{1.1}$$

The wave pressure $p=p(x, z)$ then satisfies the Helmholtz equation

$$\Delta p + \frac{\omega^2}{c^2(z)}p = 0 \tag{1.2}$$

where the wave velocity $c(z)$ is a function of the z coordinate.

Eq. (1.2) can be reduced to a second-order ordinary differential equation by a Fouriertransformation with respect to the variable x ($-\infty<x<+\infty$)

$$P(s,z) = \int_{-\infty}^{+\infty} p(x,z)\exp(isx)dx, \quad p(x,z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} P(s,z)\exp(-isx)ds \tag{1.3}$$

which, after making the replacement of variable $s=\omega\tilde{s}$ (the tilde will henceforth be omitted) takes the form

$$P''(s,z) - \omega^2\gamma^2(s,z)P(s,z) = 0, \quad \gamma(s,z) = \sqrt{s^2 - \frac{1}{c^2(z)}} \tag{1.4}$$

The prime denotes a derivative with respect to z, while the quantity s can be assumed to be a constant parameter. The branch of the quadratic root is taken such that $\mathrm{Re}(\gamma) \geq 0$.

To formulate the boundary conditions we will assume that a rigid plane vibrator of length $2a$ generates acoustic waves in a half-plane, where the amplitude of the vibrations is equal to u_0 with frequency ω. To be specific, we will assume that the vibrator is situated above the half-plane, while outside the limits of the vibrator the surface of the liquid is free from applied pressures. This leads to the following boundary conditions

$$z = 0\colon\ p(x,z) = 0, \quad |x|>a; \quad \frac{\partial p}{\partial z}(x,z) = \rho_0\omega^2 u_0, \quad |x| \leq a \tag{1.5}$$

since it follows from the defining equations of linear acoustics for the function u, representing the displacement vector, that

$$\rho_0\frac{\partial^2 u}{\partial t^2} = -\mathrm{grad}\,p \Rightarrow \rho_0\omega^2 u = \mathrm{grad}\,p \Rightarrow \rho_0\omega^2 u_z = \frac{\partial p}{\partial z} \tag{1.6}$$

where ρ_0 is the density of the medium at rest. The conditions at infinity when $z\to+\infty$ are such that uniform (i.e. non-decaying) waves must satisfy Sommerfeld's radiation condition, while non-uniform waves decay with distance.

Henceforth we will assume that the wave velocity $c(z)$ is a monotonic and bounded function of the argument $z>0$, and is at least doubly differentiable

$$c(z) \in C^2[0,\infty), \quad m \leq c(z) \leq M \quad (m, M>0)$$

(In ocean acoustics it is well known that the monotonic behaviour of $c(z)$ is characteristic for the Arctic zone to the same extent as for a mid-latitude zone in the winter.)

Henceforth we will only consider the case of a monotonically decreasing function $c(z)$.

2. Fundamental relations for the case of a monotonically decreasing wave function $c(z)$

In the case of a monotonically decreasing function $c(z)$ we put

$$c(0) = c_1, \quad c(\infty) = c_2, \quad c_1>c_2, \quad c_2 \leq c(z) \leq c_1 \tag{2.1}$$

Since the dependence on the parameter s in Eq. (1.4) is even, we will only consider non-negative values of this parameter: $0 \leq s < \infty$.

1°. Suppose

$$0 \le s < 1/c_1 \quad \text{or} \quad s > 1/c_2 \tag{2.2}$$

Then, for these values of the parameter s, the square root $\gamma(s, z) \neq 0$. In this case, for high frequencies ω, Eq. (1.4) can be solved by the classical WKB method[2–6]. We obtain

$$\begin{aligned} P(s,z) &= \frac{1}{\sqrt{\gamma(s,z)}}[A(s)\exp(-\omega\zeta(s,z)) + B(s)\exp(\omega\zeta(s,z))] \\ \zeta(s,z) &= \int_0^z \gamma(s,z)dz \end{aligned} \tag{2.3}$$

where A and B are certain as yet unknown functions.

It should be noted that the representation (2.3) holds in both cases: $s > 1/c_2 > 1/c(z)$, when $\gamma(s, z) > 0$, and $0 < s < 1/c_1 < 1/c(z)$, when

$$\gamma(s,z) = -iq(s,z), \quad q(s,z) = \sqrt{1/c^2(z) - s^2} > 0 \tag{2.4}$$

It is obvious that, in both cases, in Eq. (2.3) we must put $B(s) = 0$, since, for a non-uniform wave this gives an unbounded solution (since $\zeta(s, z) \to +\infty$ when $z \to +\infty$), while for uniform waves ($\gamma = -iq$, $q > 0$) this gives a solution with a behaviour in the far zone of the form

$$\exp(-ib\omega z), \quad z \to +\infty; \quad b = \sqrt{1/c_2^2 - s^2} > 0$$

which represents a wave, arriving from infinity, instead of a wave departing to infinity. When $A(s \neq 0, B(s) = 0$, expression (2.3) obviously satisfies Sommerfeld's radiation condition.

Hence, in the region of variation of the parameter s considered we have $B = 0$; hence

$$P(s,z) = \frac{A(s)}{\sqrt{\gamma(s,z)}}\exp(-\omega\zeta(s,z)) \Rightarrow P(s,z) = P(s,0)\sqrt{\frac{\gamma(s,0)}{\gamma(s,z)}}\exp(-\omega\zeta(s,z)) \tag{2.5}$$

2°. Suppose

$$1/c_1 < s < 1/c_2 \Rightarrow c_2 < 1/s < c_1$$

It can be seen that the equation $\gamma = 0 \Rightarrow c(z_0) = 1/s$ always has a unique solution, and this value of $z = z_0 = z_0(s)$ is a simple root of the equation $\gamma = 0$: $\gamma(s, z_0) = 0$, $\gamma'(s, z_0) \neq 0$. The point z_0 ($0 < z_0 < \infty$) is called a "point of rotation" of Eq. (1.4). It should be noted that the assumption that the function $c(z)$ is monotonic guarantees that the root z_0 is simple.

The general solution of Eq. (1.4) in this case, as is well known, looks as follows[2–6]

$$\begin{aligned} P(s,z) &= \frac{1}{\sqrt{\zeta'(s,z)}}\{A(s)\mathrm{Ai}[\omega^{2/3}\zeta(s,z)] + B(s)\mathrm{Bi}[\omega^{2/3}\zeta(s,z)]\} \\ \zeta(s,z) &= \left[\frac{3}{2}\int_{z_0}^z \gamma(s,z)dz\right]^{2/3} \end{aligned} \tag{2.6}$$

where $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ are Airy functions, while $A(x)$ and $B(x)$ are certain unknown functions.

We will investigate the behaviour of solution (2.6) at infinity. It is obvious that for $z > z_0$ we have

$$c_2 < c(z) < c(z_0) = 1/s$$

Hence, relations (2.4) are satisfied. Then

$$\zeta(s,z) \sim -\left[\frac{3}{2}q_2(s)z\right]^{2/3}, \quad z\to+\infty; \quad q_2(s) = \sqrt{\frac{1}{c_2^2}-s^2} > 0$$

This enables us to use the asymptotic form of Airy's functions for a large negative argument[7]

$$\mathrm{Ai}(-z) \sim \frac{\sin(\eta+\pi/4)}{\pi^{1/2}z^{1/4}}, \quad \mathrm{Bi}(-z) \sim \frac{\cos(\eta+\pi/4)}{\pi^{1/2}z^{1/4}}, \quad z\to+\infty; \quad \eta = \frac{2}{3}z^{3/2}$$

In the case considered

$$\eta = \omega q_2 z, \quad \zeta'(s,z) \sim -\left(\frac{2}{3}\right)^{1/3} q_2^{2/3} z^{-1/3}, \quad z\to+\infty$$

therefore

$$P(s,z) \sim \mathrm{const}(s)[A(s)\sin(\omega q_2 z+\pi/4) + B(s)\cos(\omega q_2 z+\pi/4)], \quad z\to\infty \tag{2.7}$$

This solution must satisfy the radiation condition, i.e. it must have the form

$$P(s,z) \sim \mathrm{const}(s)\exp(idz), \quad z\to+\infty; \quad d>0$$

This can obviously only occur when

$$A(s) = iB(s)$$

Thus, the general solution of the equation in the case considered, which satisfies the radiation condition at infinity, has the form

$$P(s,z) = P(s,0)\sqrt{\frac{\zeta'(s,0)}{\zeta'(s,z)}}\frac{Q(s,z)}{Q(s,0)}; \quad Q(s,z) = i\mathrm{Ai}[\omega^{2/3}\zeta(s,z)] + \mathrm{Bi}[\omega^{2/3}\zeta(s,z)] \tag{2.8}$$

3. The fundamental integral equation

To satisfy boundary conditions (1.5) we will calculate the derivative of the solution (2.5), (2.8). It is obvious that the main term of the asymptotic form is obtained when the derivative is applied solely to the Airy functions in Eq. (2.8) and only to the exponential function in Eq. (2.5), and hence, for high frequencies we have

$$P'(s,z) = -\omega P(s,0)\sqrt{\frac{\gamma(s,0)}{\gamma(s,z)}}\zeta'(s,z)\omega\exp(-\omega\zeta(s,z)), \quad s\in\left[0,\frac{1}{c_1}\right)\cup\left(\frac{1}{c_2},\infty\right) \tag{3.1}$$

$$P'(s,z) = \omega^{2/3}P(s,0)\sqrt{\zeta'(s,0)\zeta'(s,z)}\frac{Q'(s,z)}{Q(s,0)}, \quad s\in\left(\frac{1}{c_1},\frac{1}{c_2}\right) \tag{3.2}$$

Bearing in mind the explicit expressions for the functions $\zeta(s,z)$ (2.3) and $\zeta(s,z)$ (2.6), we obtain

$$P'(s,0) = -\omega P(s,0)\gamma_1(s), \quad \gamma_1(s) = \sqrt{s^2-\frac{1}{c_1^2}}, \quad s\in\left[0,\frac{1}{c_1}\right)\cup\left(\frac{1}{c_2},\infty\right) \tag{3.3}$$

$$P'(s,0) = -\omega^{2/3}P(s,0)\zeta^{-1/2}(s,0)\gamma_1(s)\frac{Q'(s,0)}{Q(s,0)}, \; s\in\left(\frac{1}{c_1},\frac{1}{c_2}\right), \; \zeta(s,0) = \left[\frac{3}{2}\int_0^{z_0}\gamma(s,z)dz\right]^{2/3} \tag{3.4}$$

Note that when $s\in(1/c_1, 1/c_2)$ and $0\le z<z_0$ we have $c(z)>c(z_0)=1/s$, i.e. $\gamma>0$, $\gamma_1>0$.

It is now clear that we can apply an asymptotic expansion of the Airy functions to expression (3.4) for a large positive argument[7]

$$\mathrm{Ai}(z) \sim \frac{\exp(-\eta)}{2\pi^{1/2}z^{1/4}}, \quad \mathrm{Bi}(z) \sim \frac{\exp(\eta)}{2\pi^{1/2}z^{1/4}}, \quad z \to +\infty; \quad \eta = \frac{2}{3}z^{2/3}$$

Consequently, expressions (3.4) and (3.3) are asymptotically equivalent, and hence

$$P'(s,0) = -\omega P(s,0)\gamma_1(s), \quad 0 \le s < \infty, \tag{3.5}$$

and this relation can be extended, as was stated above, over the whole axis $|s| < \infty$ by virtue of the obvious evenness.

Applying an inverse Fourier transformation to relation (3.5) and taking into account the boundary conditions (1.5), we obtain an integral equation with a kernel of the convolution type

$$\int_{-a}^{+a} g(\xi)K(x-\xi)d\xi = -\rho_0 u_0, \ |x| \le a; \ g(x) = p(x,0); \ K(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(ixs\omega)\gamma_1(s)ds \tag{3.6}$$

(we recall that $s = \tilde{s}\omega$ and the tilde sign is omitted).

Thus, as a result we obtain the same integral equation known from the classical theory for an acoustic half-plane with a constant wave velocity. This classical case also follows directly from the above discussion, since when $c(z) \equiv c$ we have $c_1 = c_2 = c$, and it all reduces to the case (3.3), which is identical with the well-known solution for a uniform half-plane.

It is also obvious that when $c(z) \equiv c$, formula (2.5) gives an exact solution of Eq. (1.4).

4. Some properties of hypersingular integrals and of the fundamental integral equation

Note that the integral which defines the kernel in Eq. (3.6) diverges in the classical sense. In fact,

$$\begin{aligned} K(x) &= \frac{1}{\pi}\int_0^\infty \sqrt{s^2 - \frac{1}{c_1^2}}\cos(x\omega s)ds = \\ &= \frac{1}{\pi}\int_0^\infty s\cos(x\omega s)ds + \frac{1}{\pi}\int_0^\infty \left(\sqrt{s^2 - \frac{1}{c_1^2}} - s\right)\cos(x\omega s)ds = K_0(x) + K_1(x) \end{aligned} \tag{4.1}$$

The integral $K_1(x)$ converges in the classical sense

$$K_1(x) = -\frac{1}{\pi c_1^2}\int_0^\infty \frac{\cos(x\omega s)}{\sqrt{s^2 - 1/c_1^2} + s}ds \tag{4.2}$$

since the integrand is the product of a certain monotonically decreasing function of the order of $1/s$ ($s \to +\infty$) and an oscillating function. The integral $K_0(x)$ can be considered in a generalized sense[8] (everywhere henceforth the limit is taken as $\varepsilon \to +0$)

$$\begin{aligned} K_0(x) &= \frac{1}{\pi}\int_0^\infty s\cos(\omega x s)ds = \frac{1}{\pi}\lim\int_0^\infty \exp(-\varepsilon s)s\cos(\omega x s)ds = \\ &= \frac{1}{\pi}\lim\frac{\varepsilon^2 - \omega^2 x^2}{(\varepsilon^2 + \omega^2 x^2)^2} = -\frac{1}{\pi\omega^2 x^2} \end{aligned} \tag{4.3}$$

Hence, the kernel $K(x)$ has hypersingular behaviour as $x \to 0$.

It is first necessary to clarify in what sense one can treat hypersingular integrals, since they do not exist either as improper integrals of the first kind, or as Cauchy-type singular integrals. At least three different definitions of hypersingular equations are known, namely,

1) the integral is the derivative of the principal Cauchy value

$$\int_a^b \frac{\varphi(t)}{(x-t)^2}dt = -\frac{d}{dt}\int_a^b \frac{\varphi(t)}{x-t}dt \tag{4.4}$$

2) the integral is treated as a Hadamard principal value

$$\int_a^b \frac{\varphi(t)}{(x-t)^2}dt = \lim\left[\left(\int_a^{x-\varepsilon} + \int_{x+\varepsilon}^b\right)\frac{\varphi(t)}{(x-t)^2}dt - \frac{2\varphi(x)}{\varepsilon}\right] \tag{4.5}$$

3) the integral is a "residue" in the sense of generalized functions, i.e. it is the analytic continuation of the integral

$$\int_a^b |x-t|^\alpha \varphi(t)dt \tag{4.6}$$

with respect to the parameter α.

All three definitions are equivalent if the density $\varphi(x)$ is an analytic function (at least twice continuously differentiable) in the open interval (a, b).

In fact, suppose $\varphi(x)$ is an analytic function and $\mathrm{Re}(\alpha) > 1$. Then

$$-\frac{d}{dx}\int_a^b \frac{\varphi(t)}{x-t}dt = -\lim\frac{d}{dx}\left(\int_a^{x-\varepsilon} + \int_{x+\varepsilon}^b\right)\frac{\varphi(t)}{x-t}dt = \lim\left[\left(\int_a^{x-\varepsilon} + \int_{x+\varepsilon}^b\right)\frac{\varphi(t)}{(x-t)^2}dt - \frac{2\varphi(x)}{\varepsilon}\right]$$

$$\int_a^b |x-t|^\alpha \varphi(t)dt = \lim\left(\int_a^{x-\xi} (x-t)^\alpha\varphi(t)dt + \int_{x+\varepsilon}^b (t-x)^\alpha\varphi(t)dt\right) =$$

$$= \frac{d}{dx}\lim\left(-\int_x^{x-\varepsilon} \frac{(x-t)^{\alpha+1}}{\alpha+1}\varphi(t)dt + \int_{x+\varepsilon}^b \frac{(t-x)^{\alpha+1}}{\alpha+1}\varphi(t)dt\right)$$

Hence, applying analytic continuation to the last relation, using the standard (ε, δ) formalism one can verify the correctness of equality (4.4). Consequently, the equivalence of definitions 1, 2 and 3 is obvious.

If $\varphi(x) \in C_2(a, b)$, a finite value of the limit in expression (4.5) exists, since the expression in square brackets in relation (4.5) is equal to

$$\left(\int_a^{x-\varepsilon} + \int_{x+\varepsilon}^b\right)\frac{\varphi(t)-\varphi(x)-\varphi'(x)(t-x)}{(x-t)^2}dt + \varphi(x)\frac{a-b}{(x-a)(b-x)} + \varphi'(x)\ln\frac{b-x}{x-a}$$

and has a finite limit as $\varepsilon \to +0$. Hence, for $x \in (a, b)$ the integral on the left-hand side of equality (4.5) is finite in any sense.

Theorem 1. *If* Eq. (3.6) *has a bounded solution* $g(x), x \in -[a, a]$, *then* $g(x) \in C_1(-a, a)$ *and* $g(x) \sim \sqrt{a \pm x}$, $x \to \mp a$.

Proof. We first note that the solution of the characteristic hypersingular equation

$$\int_{-a}^a \frac{\varphi(t)dt}{(x-t)^2} = f'(x), \quad |x| \le a$$

can be constructed by integration using equality (4.4). We obtain

$$\int_{-a}^b \frac{\varphi(t)}{x-t}dt = -f(x) + C, \quad |x| \le a$$

(C is an arbitrary constant). Inversion of the singular Cauchy integral leads to a single bounded solution in the form[9]

$$\varphi(x) = \frac{\sqrt{a^2-x^2}}{\pi^2}\int_{-a}^{a}\frac{f(t)}{\sqrt{a^2-t^2}(x-t)}dt, \quad |x| \le a$$

where

$$C = \frac{1}{\pi^2}\int_{-a}^{a}\frac{f(t)}{\sqrt{a^2-t^2}}dt$$

We apply this result to the complete equation (3.6), transferring the regular part of the kernel to the right-hand side of the equality

$$-\frac{1}{\pi\omega^2}\int_{-a}^{a}\frac{g(\xi)}{(x-\xi)^2}d\xi = -\rho_0 u_0 + \frac{1}{\pi c_1^2}\int_{-a}^{a}g(\xi)d\xi\int_0^{\infty}\frac{\cos[(x-\xi)\omega s]}{\sqrt{s^2-1/c_1^2}+s}ds$$

Inverting the characteristic part of the kernel, we obtain the relations

$$g(x) = \frac{\sqrt{a^2-x^2}}{\pi^2}\int_{-a}^{a}\frac{\pi g_0 u_0\omega^2 t-(\omega/c_1)^2 f(t)}{\sqrt{a^2-t^2}(x-t)}dt$$

$$f(t) = \frac{1}{\omega}\int_{-a}^{a}g(\xi)d\xi\int_0^{\infty}\frac{\sin[(t-\xi)\omega s]}{s(\sqrt{s^2-1/c_1^2}+s)}ds$$

from which the assertion of the theorem follows. □

Theorem 2. *If* Eq. (3.6) *has a bounded solution* $g(x)$, $x \in -[a, a]$, *this solution is unique.*

Proof. If two solutions $u_1(x)$ and $u_2(x)$ exists, they both belong to the class established by the previous theorem. Their difference u_Δ satisfies the homogeneous equation

$$\int_{-a}^{a}u_\Delta(\xi)K(x-\xi)d\xi = 0, \quad |x| \le a$$

By calculating the scalar product of the last equality with the function $u_\Delta(x)$ and using Parseval's equality, we arrive at the relation

$$\int_{-\infty}^{\infty}\gamma_1(s)|\mathcal{U}_\Delta(s)|^2 ds = 0$$

whence

$$\left(\int_{\infty}^{-1/c_1} + \int_{1/c_1}^{\infty}\right)\sqrt{s^2-1/c_1^2}|\mathcal{U}_\Delta(s)|^2 ds - i\int_{-1/c_1}^{1/c_1}\sqrt{1/c_1^2-s^2}|\mathcal{U}_\Delta(s)|^2 ds = 0 \tag{4.7}$$

Here $U_\Delta(s) = U_1(s) - U_2(s)$ is the Fourier transform of the function $u_\Delta(x)$. Equality (4.7) is only possible if $U_1(s) \equiv U_2(s)$, $|s| < +\infty$, i.e. if $u_1(x) \equiv u_2(x)$, $|x| \le a$. Note that the first two integrals on the left-hand side of equality (4.7) are finite; this follows from the behaviour of the integrand and Erdelyi's lemma, according to which

$$\int_0^A x^{\beta-1}f(x)e^{i\lambda x}dx \sim \frac{\Gamma(\beta)}{(-i\lambda)^\beta}, \quad \lambda \to +\infty\ (\beta > 0), \quad f(x) \in C_1[0, A], \quad f(A) = 0 \tag{4.8}$$

In the case considered $\beta = 3/2$; consequently $U_1(s) = U_2(s) = O(1/s^{3/2})$, $s \to \infty$, and the integrals in relation (4.7) converge. □

5. The high-frequency asymptotic solution of Eq. (3.6)

We will now consider the construction of the high-frequency asymptotic solution of hypersingular Eq. (3.6). The approach used follows the general idea of Aleksandrov,[10] being an alternative to the classical edge-wave method.[11]

Putting

$$g(x) = \upsilon(a+x) + \upsilon(a-x) - w(x), \quad |x| \le a$$

we conclude that Eq. (3.6) is equivalent to a pair of equations in the new unknown functions $\upsilon(x)$ and $w(x)$

$$\int_0^\infty \upsilon(\xi)K(x-\xi)d\xi = -\rho_0 u_0 + I(x), \ 0 < x < \infty; \ I(x) = \int_0^\infty [\upsilon(2a+\xi) - w(\xi)]K(x+\xi)d\xi \tag{5.1}$$

$$\int_{-\infty}^\infty w(\xi)K(x-\xi)d\xi = -\rho_0 u_0, \quad -\infty < x < +\infty \tag{5.2}$$

The basic idea of the "small parameter" method[10] is to construct a solution of Eq. (5.1) with a neglected integral $I(x)$. This approach reduces this equation to a Wiener-Hopf equation, solvable in explicit form.[1,13] This method is obviously only correct in those problems in which it is possible to prove that the explicit solution constructed, on being substituted into the function $I(x)$, gives this function a value that is asymptotically small. In the problem considered, this approach will only be correct if the solution constructed satisfies the condition

$$I(x) \to 0, \quad \omega \to \infty \tag{5.3}$$

which it is necessary to verify separately.

In order to estimate the integral $I(x)$, we will estimate the asymptotic form of the kernel $K(x)$ at infinity. Making cuts in the complex plane of the variable s, necessary to distinguish the unique branch of the square root of $\gamma_1(s)$, after a number of transformations we arrive at an asymptotic estimate, which holds for high frequencies (compare with relation (4.8)),

$$K(x) = O((\omega x)^{-3/2}), \quad \omega \to \infty \tag{5.4}$$

Further, we make the replacement of variables $x = x'/\omega$, $\xi = \xi'/\omega$ in relations (5.1)

$$K(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ixs}\gamma_1(s)ds, \quad \int_0^\infty \upsilon\left(\frac{\xi}{\omega}\right)K(x-\xi)d\xi = -\rho_0 u_0 \omega, \quad x > 0 \tag{5.5}$$

The prime is omitted here.

The formalism of the Wiener-Hopf method[12,13] now gives

$$V_+(s)\sqrt{s^2 - \frac{1}{c_1^2}} = \left(\frac{\rho_0 u_0 \omega}{is}\right)_+ + F_-(s) \tag{5.6}$$

where the plus and minus subscripts relate to functions that are analytic in the upper and lower complex half-planes respectively. Factorization of the square root on the left-hand side ofEq. (5.6) is simple:

$$\sqrt{s^2 - \frac{1}{c_1^2}} = \left(\sqrt{s + \frac{1}{c_1}}\right)_+ \left(\sqrt{1 - \frac{1}{c_1}}\right)_-$$

Hence, Eq. (5.6) is equivalent to the following equation

$$V_+(s)\left(\sqrt{s+\frac{1}{c_1}}\right)_+ = \rho_0 u_0 \omega\left(\frac{1}{is}\right)_+\left(\sqrt{s-\frac{1}{c_1}}\right)_-^{-1} + N_-(s)$$

The expansion of the first function on the right-hand side is also quite obvious:

$$\left(\frac{1}{is}\right)_+\left(\sqrt{s-\frac{1}{c_1}}\right)_-^{-1} = \left\{\left[\left(\sqrt{s^2-\frac{1}{c_1^2}}\right)^{-1} - \left(\sqrt{-\frac{1}{c_1}}\right)^{-1}\right]\frac{1}{is}\right\}_- + \left[\left(\sqrt{-\frac{1}{c_1}}\right)^{-1}\frac{1}{is}\right]_+$$

which leads to the relation

$$V_+(s)\left(\sqrt{s+\frac{1}{c_1}}\right)_+ + \rho_0 u_0 \omega\left(\frac{\sqrt{-c_1}}{is}\right)_+ = M_-(s) \tag{5.7}$$

Here, on the left-hand side, we have a function that is analytic in the upper half-plane of the complex variable s, while on the right-hand side we have a function that is analytic in the lower half-plane. According to the principle of analytic continuation, if these functions are identical, at least along the real axis $\mathrm{Im}(s) = 0$, they represent the same integral function. We will estimate the behaviour of this function at infinity ($s \to +\infty$).

We will seek a solution of the problem in the class of pressures $g(x)$, limited along the base of the vibrator $|x| \le a$. Its Fourier transform is then $V_+(x) = O\ 1/s,\ s \to +\infty$. Consequently, the left-hand side of Eq. (5.7), representing the integral function in the upper half-plane, approaches zero as $s \to +\infty$, and hence this integral function is identical zero, which leads to the relation

$$V_+(s)\sqrt{s+\frac{1}{c_1}} + \rho_0 u_0 \omega\frac{\sqrt{-c_1}}{is} = 0 \Rightarrow V_+(s) = \rho_0 u_0 \omega i\frac{\sqrt{c_1}}{s\sqrt{s+\frac{1}{c_1}}}$$

The original of this Fourier transform can be found from tables:[14]

$$\upsilon(x) = -\rho_0 u_0 \omega i c_1 \operatorname{erf}\left(\sqrt{-\frac{ix}{c_1}}\right) = -\sqrt{2}\rho_0 u_0 \omega c_1 e^{\pi i/4}\left[C\left(\frac{x}{c_1}\right) + iS\left(\frac{x}{c_1}\right)\right] \tag{5.8}$$

where $\operatorname{erf}(x)$ is the probability integral and $C(x)$ and $S(x)$ are Fresnel integrals.[7]

Now, when the asymptotic solution of the problem is constructed, it is necessary to verify that the asymptotic estimate (5.3) is correct. The explicit representation (5.8) for $\upsilon(x)$ and the asymptotic estimate of the kernel (5.4) prove that relation (5.3) is true.

Finally, the equation in convolutions is easily solved by using a Fourier transformation, which leads to the expression

$$W(s)\sqrt{s^2-\frac{1}{c_1^2}} = -2\pi\rho_0 u_0 \omega \delta(s)$$

where δ is the Dirac delta function. Hence, we obtain

$$w(x) = -\rho_0 u_0 \omega c_1 i \tag{5.9}$$

In conclusion we point out that the solution of the fundamental integral equation constructed defines the function $P(s, 0)$. Consequently, the total-pressure field in the medium can be found from relations (2.5) and (2.8).

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